Discrete Mathematics: Combinatorics and Graph Theory

Practice Exam 1 Solution

Instructions. Solve any 5 questions. Write neatly and show your work to receive full credit. Note that this is a sample exam and while it bears some similarity with the real exam, the two are not isomorphic.

1. Prove the following by induction. State base case, inductive hypothesis and inductive step explicitly.

   (a) $\sum_{i=1}^{n} 2^{i-1} = \sum_{i=0}^{n-1} 2^i = 2^n - 1$

   Base case when $n = 1 : 2^{1-1} = 1 \equiv 2^1 - 1 = 1$.
   Inductive hypothesis: Assume $\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$ true for some k.
   Inductive step:

$$\begin{aligned}\sum_{i=1}^{k+1} 2^{i-1} &= \underbrace{(\sum_{i=1}^{k} 2^{i-1})}_{2^k-1 \text{ by the IH}} + 2^{k+1-1} \\ &= 2^k - 1 + 2^k \\ &= 2^{k+1} - 1 \quad \square\end{aligned}$$

   (b) $\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$

   Base case when $n = 1 : 1(2^1) = 2 \equiv 2 + (1-1)2^{1+1} = 2$.
   Inductive hypothesis: Assume $\sum_{i=1}^{k} i(2^i) = 2 + (k-1)2^{k+1}$ true for some k.
   Inductive step:

$$\begin{aligned}\sum_{i=1}^{k+1} i(2^i) &= \underbrace{\sum_{i=1}^{k} i(2^i)}_{2+(k-1)2^{k+1} \text{ by the IH}} + (k+1)(2^{k+1}) \\ &= (2^{k+1})\left[(k-1) + (k+1)\right] + 2 \\ &= (2^{k+1})2k + 2 \\ &= 2 + k \times 2^{k+2} \quad \square\end{aligned}$$

   (c) $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$

   Base case when $n = 1 : 1(1!) = 1 \equiv (1+1)! - 1 = 2 - 1 = 1$.
   Inductive hypothesis: Assume $\sum_{i=1}^{k} i(i!) = (k+1)! - 1$ true for some k.
   Inductive step:

$$\begin{aligned}\sum_{i=1}^{k+1} i(i!) &= \underbrace{\sum_{i=1}^{k} i(i!)}_{(k+1)!-1 \text{ by the IH}} + (k+1)(k+1)! \\ &= (k+1)! - 1 + (k+1)(k+1)! \\ &= (k+1)!(1 + k + 1) - 1 \\ &= (k+2)(k+1)! - 1 \\ &= (k+2)! - 1 \quad \square\end{aligned}$$

2. Prove the following:

   (a) The sum of any three consecutive integers is divisible by 3.

   $$n + (n+1) + (n+2) = 3n + 3 = 3(n+3) \Rightarrow \exists x \in \mathbb{Z} \text{ such that } 3x = 3(n+3) \Rightarrow 3|3(n+3) \quad \square$$

   (b) For all integers a, b, and c, if a divides b and a divides c then a divides b + c.
   If $a|b \Rightarrow \exists\ r \in \mathbb{Z}$ such that $ar = b$. Similarly if $a|c \Rightarrow \exists\ s \in \mathbb{Z}$ such that $as = c$. Then $b + c = ar + as = a(r+s) \Rightarrow \exists\ k \in \mathbb{Z}$ such that $ak = bc \Rightarrow a|a(r+s) \quad \square$.

   (c) Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 1$. Hint: use proof by contradiction.
   Assume there exists a positive integer solution. Then $(x-y)(x+y) = 1$. This implies that either (i): $(x-y) = 1$ and $(x+y) = 1$ or (ii): $(x-y) = -1$ and $(x+y) = -1$. Adding the two equations in (i) $\Rightarrow 2x = 2 \Rightarrow x = 1 \Rightarrow y = 0$ and adding the two equations in (ii) $2x = -2 \Rightarrow x = -1 \Rightarrow y = 0$. In either case we have a contradiction since $y = 0 \notin \mathbb{Z}^+ \quad \square$.

3. Prove the following by induction. State base case, inductive hypothesis and inductive step explicitly.

   (a) For all $n \in \mathbb{Z}^+, n > 3 \Rightarrow 2^n < n!$
   Base case when $n = 4 : 2^4 = 16 > 4! = 24$.
   Inductive hypothesis: $2^k < k!$ for some $k > 3$.
   Inductive step:

   $$\begin{aligned} 2^{k+1} &= 2 \times 2^k \\ &< 2 \times k! \text{ by the IH} \\ &< (k+1) \times k! \text{ (since } k > 3 \Rightarrow k+1 > 2) \quad \square \end{aligned}$$

   (b) For all $n \in \mathbb{Z}^+, n > 4 \Rightarrow n^2 < 2^n$
   Base case when $n = 5 : 5^2 = 25 < 2^5 = 32$.
   Inductive hypothesis: Assume $k^2 < 2^k$ for some $k > 4$.
   Inductive step:

   $$\begin{aligned} (k+1)^2 &= k^2 + 2k + 1 \\ &< 2^k + 2k + 1 \text{ by the IH} \\ &< 2^k + 2^k \text{ (since } k > 4 \Rightarrow 2^k > 2k+1) \\ &= 2^{k+1} \quad \square \end{aligned}$$

   (c) For all $n \in \mathbb{N}$, $3|(7^n - 4)$
   Base case when $n = 1 : 3|(7^1 - 4)$.
   Inductive hypothesis: Assume $3|(7^k - 4)$ for some k.
   Inductive step:
   Rewording the IH: $3|(7^k - 4) \Rightarrow \exists x \in \mathbb{Z}$ such that $3x = 7^k - 4 \Rightarrow 7^k = 3x - 4$.

   $$\begin{aligned} 7^{k+1} - 4 &= 7 \times 7^k - 4 \\ &= 7(3x-4) - 4 \text{ by the IH} \\ &= 21x - 24 \\ &= 3(7x - 8) \Rightarrow 3|(7^{k+1} - 4) \quad \square \end{aligned}$$

4. Verify whether the following functions define bijections.

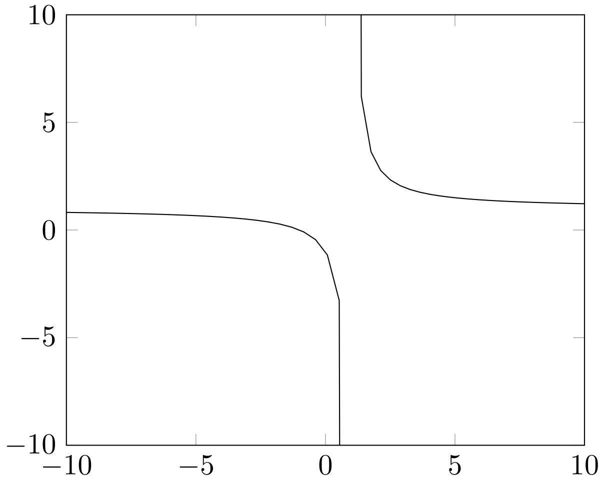

Figure 1: Visualizing the function $f(x) = \frac{x+1}{x-1}$. There are two asymptotes at $x = 1$ and $y = 1$ which imply that the function is not onto for $Y = \mathbb{R}$.

(a) Let $m \neq 0$ and b be real numbers. Is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = mx + b$ (i) injective (ii) surjective and (iii) bijective? Prove or disprove.
Recall the definition of an injective function: $\forall x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Applying to our function: $f(x_1) = mx_1 + b$ and $f(x_2) = mx_2 + b$. Therefore $f(x_1) = f(x_2) \Rightarrow mx_1 + b = mx_2 + b \Rightarrow mx_1 = mx_2 \Rightarrow x_1 = x_2$. Therefore $f(x)$ is one-to-one.
Recall the definition of a surjective function: $\forall y \in Y, \exists x \in X$ such that $f(x) = y$. Applying to our function: $y = mx + b \Rightarrow y - b = mx \Rightarrow x = {}^{y-b}\!/_{m}$. Taking the image of x under f yields $f(x) = m({}^{y-b}\!/_{m}) + b = y - b + b = y$. Therefore $f(x)$ is onto.

(b) Let $S = \{x \in \mathbb{R} : x \neq 1\}$. Is the function $f : S \to \mathbb{R}$ defined by $f(x) = \frac{x+1}{x-1}$ (i) injective (ii) surjective and (iii) bijective? Prove or disprove.

$$\begin{aligned}
f(x_1) = f(x_2) &\Rightarrow \frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1} \\
&\Rightarrow (x_1+1)(x_2-1) = (x_2+1)(x_1-1) \\
&\Rightarrow x_1x_2 - x_1 + x_2 - 1 = x_1x_2 - x_2 + x_1 - 1 \\
&\Rightarrow 2x_2 = 2x_1 \\
&\Rightarrow x_2 = x_1
\end{aligned}$$

Therefore $f(x)$ is one-to-one.
To check whether $f(x)$ is surjective, we attempt to take the inverse:

$$\begin{aligned}
y &= \frac{x+1}{x-1} \\
y(x-1) &= x+1 \\
yx - y &= x+1 \\
yx - x &= y+1 \\
x(y-1) &= y+1
\end{aligned}$$

We would like to solve to x in terms of y, however the codomain $\mathbb{R}$ includes the number 1. We therefore cannot divide by $y - 1$ since this operation is undefined. Hence it is not true that $\forall y \in Y, \exists x \in X$ such that $f(x) = y$. Therefore, $f(x)$ is not surjective.

5. Use the Euclidean Algorithm to find the greatest common divisor of 44 and 18. Use the Extended Euclidean Algorithm to find the Bezout coefficients x, y and all integer solutions to the equation $44x + 18y = gcd(44, 18)$.

$$44 = 2 \times 18 + 8 \quad \Rightarrow \quad 8 = 44 - 2 \times 18 \tag{1}$$
$$18 = 2 \times \ 8 + 2 \quad \Rightarrow \quad 2 = 18 - 2 \times 8 \tag{2}$$
$$8 = 4 \times 2 + 0 \tag{3}$$

We can now back substitute, expressing each remainder as a linear combination of 44 and 18 starting with Equation 2:

$$\begin{aligned} 2 &= 18 - 2 \times 8 \\ &= 18 - 2 \times (44 - 2 \times 18) \\ &= 5 \times 18 - 2 \times 44 \end{aligned}$$

Therefore the Bezout coefficients are $x = -2$ and $y = 5$. Using the identity $lcm(a, b) = ab/gcd(a, b)$:

$$\frac{44 \times 18}{2} \Rightarrow 44 \times \frac{18}{2} = 18 \times \frac{44}{2} \tag{4}$$
$$\Rightarrow 44 \times 9 + 18 \times -22 = 0 \tag{5}$$

Multiply through by an integer k:

$$44 \times 9k + 18 \times -22k = 0 \tag{6}$$

Using Bezout's identity:

$$44 \times 2 + 18 \times 5 = 2 \tag{7}$$

Adding the two identities in Equation 6 and Equation 7 yields an infinite set of solutions:
$\Rightarrow 44(9k - 2) + 18(-22k + 5) = 2$.

6. The Fermat numbers are defined as $f(n) = 2^{2^n} + 1$. Prove that for all $n \geq 0$, $f_{n+1} = f_0 \times f_1 \times \cdots \times f_n + 2$. Let's work out the first few Fermat numbers:

$$\begin{aligned} f_0 &= 2^{2^0} + 1 = 2^1 + 1 = 3 \\ f_1 &= 2^{2^1} + 1 = 2^2 + 1 = 5 \\ f_2 &= 2^{2^2} + 1 = 2^4 + 1 = 17 \end{aligned}$$

Base case when $n = 0 : f_{0+1} = f_1 = 5 \equiv f_0 = 2 = 3 + 2$.
Inductive hypothesis: Assume $f_{k+1} = f_0 \times f_1 \times \cdots \times f_k + 2$ true for some k.
Inductive step:
Let's rewrite the inductive hypothesis as follows to isolate the product $(f_0 f_1 f_2 \cdots f_k)$:
$f_{k+1} = (f_0 f_1 f_2 \cdots f_k) + 2 \Rightarrow f_0 f_1 f_2 \cdots f_k = f_{k+1} - 2$

$$\begin{aligned} f_0 f_1 f_2 \cdots f_k) f_{k+1} + 2 &= (f_{k+1} - 2) f_{k+1} + 2 \\ &= (f_{k+1}^2 - 2f_{k+1} + 1) + 1 \\ &= (f_{k+1} - 1)^2 + 1 \\ &= (2^{2^{k+1}} + 1 - 1)^2 + 1 \\ &= 2^{2^{k+2}} + 1 \\ &= f_{k+2} \quad \square \end{aligned}$$